

# Lonesum decomposable matrices

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## Abstract

A lonesum matrix is a  $(0, 1)$ -matrix that is uniquely determined by its row and column sum vectors. In this paper, we introduce lonesum decomposable matrices and study their properties. We provide a necessary and sufficient condition for a matrix  $A$  to be lonesum decomposable, and give a generating function for the number  $D_k(m, n)$  of  $m \times n$  lonesum decomposable matrices of order  $k$ . Moreover, by using this generating function we prove some congruences for  $D_k(m, n)$  modulo a prime.

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## 1 Introduction

A  $(0, 1)$ -*matrix* (resp. *vector*) is a matrix (resp. vector) in which each entry is zero or one. A  $(0, 1)$ -matrix  $A$  is called a *lonesum matrix* if  $A$  is uniquely determined by its row and column sum vectors. For example, a  $(0, 1)$ -matrix with a row sum vector  ${}^t(3, 1)$  and a column sum vector  $(1, 2, 1)$  is uniquely determined as the following:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, the matrix  $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$  is a lonesum matrix. Because  $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$  have the same row and column sum vectors, they are not lonesum matrices. We denote by  $L(m, n)$  the number of  $m \times n$  lonesum matrices. For simplicity, we set  $L(m, 0) = L(0, m) = 1$  for any non-negative integer  $m$ . It is known that lonesum matrices are related to certain combinatorial objects. For example, the number  $L(m, n)$  is equal to the number of acyclic orientations of the complete bipartite graph  $K_{m,n}$  ([5, Theorem 2.1]).

An  $m \times n$   $(0, 1)$ -matrix  $A = (a_{ij})$  is called a *Ferrers matrix* if  $A$  satisfies the condition

$$\begin{cases} a_{ij} = 0 \Rightarrow a_{kj} = 0 & (k \geq i), \\ a_{ij} = 0 \Rightarrow a_{il} = 0 & (l \geq j). \end{cases}$$

This condition means that all 1 entries of  $A$  are placed to the upper left of  $A$ . For example, the matrix  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$  is a Ferrers matrix. Ryser [12] investigated matrices that have fixed row and column sum vectors. In our setting, his result can be written as follows:

**Proposition 1.1.** *Let  $A$  be a  $(0, 1)$ -matrix. Then, the following conditions are equivalent:*

- (i)  $A$  is a lonesum matrix.
- (ii)  $A$  does not contain  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  as a submatrix.
- (iii)  $A$  is obtained from a Ferrers matrix by permutations of rows and columns.

For an integer  $k$ , Kaneko [10] introduced poly-Bernoulli numbers  $B_n^{(k)}$  of index  $k$  as

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad (1)$$

where  $\text{Li}_k(z)$  denotes the  $k$ -th polylogarithm, defined by  $\text{Li}_k(z) := \sum_{n=1}^{\infty} z^n / n^k$ . Brewbaker [4] proved that the numbers  $L(m, n)$  are equal to the poly-Bernoulli numbers of negative indices:

$$L(m, n) = B_n^{(-m)} \quad (m, n \geq 0). \quad (2)$$

The generating function of poly-Bernoulli numbers of negative indices has been given by Kaneko [10], hence the numbers of lonesum matrices have the following generating function:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L(m, n) \frac{x^m}{m!} \frac{y^n}{n!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \quad (3)$$

The present author, Ohno, and Yamamoto [9] introduced “weighted” lonesum matrices and a simple proof of (3) was given (see [9, Proof of Theorem 1]).

For  $m \times n$  matrices  $A$  and  $B$ , we write  $A \sim B$  if  $A$  is obtained from  $B$  by row or column exchanges. We call a  $(0, 1)$ -matrix  $A$  is *lonesum decomposable* if  $A$  satisfies the condition

$$A \sim \begin{pmatrix} L_1 & & & O \\ & L_2 & & \\ & & \ddots & \\ O & & & L_k \end{pmatrix},$$

where  $L_i$  ( $1 \leq i \leq k$ ) are lonesum matrices. A lonesum matrix is clearly lonesum decomposable. Since a lonesum matrix can be obtained from a Ferrers matrix, a lonesum decomposable matrix  $A$  can be transformed as

$$A \sim \begin{pmatrix} F_1 & & & O \\ & F_2 & & \\ & & \ddots & \\ O & & & F_k \\ & & & & O \end{pmatrix}, \quad (4)$$

where  $F_i$  ( $1 \leq i \leq k$ ) are Ferrers matrices with no zero rows or zero columns. We call the right-hand side of (4) the *decomposition matrix* of  $A$  and  $k$  the *decomposition order* of  $A$ .

**Proposition 1.2.** *Let  $A$  be a lonesum decomposable matrix. Then the decomposition matrix of  $A$  is uniquely determined up to the order of  $F_i$  ( $1 \leq i \leq k$ ). In particular, the decomposition order of  $A$  is uniquely determined.*

*Proof.* For a lonesum decomposable matrix  $A = (a_{ij})$ , it follows from Proposition 1.1 that two elements  $a_{ij} = 1$  and  $a_{i'j'} = 1$  belong to the same Ferrers block if and only if  $a_{ij}$  and  $a_{i'j'}$  do not form a submatrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Because the type of Ferrers matrix is uniquely determined, a decomposition matrix of  $A$  is also uniquely determined up to the order of its Ferrers blocks.  $\square$

The outline of this paper is as follows. In Section 2, we show that a  $(0, 1)$ -matrix  $A$  is lonesum decomposable if and only if  $A$  does not contain certain matrices as submatrices. In Section 3, we give a generating function for the number of lonesum decomposable matrices. In Section 4, we derive some congruences for the numbers of lonesum decomposable matrices of order  $k$  by using the generating function given in Section 3.

## 2 Lonesum decomposable matrices

Let us define a  $2 \times 3$  matrix  $U$  as

$$U := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It can be easily checked that  $U$  is not lonesum decomposable. Let  $\mathcal{N}$  be a set of all matrices obtained from  $U$  or  ${}^tU$  by permutations of rows and columns. Namely, the elements of  $\mathcal{N}$  are the following twelve matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The following is the first main result of this paper.

**Theorem 2.1.** *Let  $A$  be a  $(0, 1)$ -matrix. Then, the following two conditions are equivalent.*

- (i)  $A$  is lonesum decomposable.
- (ii)  $A$  does not contain an element of  $\mathcal{N}$  as a submatrix.

*Proof.* It is clear that (i)  $\Rightarrow$  (ii), and we show (ii)  $\Rightarrow$  (i). This statement clearly holds for  $0 \leq m, n \leq 2$ , where  $m$  and  $n$  are the numbers of rows and columns of  $A$ , respectively. A transpose of a lonesum decomposable matrix is also lonesum decomposable, hence we only have to prove that if the statement holds for all  $m \times n$  matrices, then it holds for any  $m \times (n+1)$  matrix for  $m, n \geq 2$ .

Let  $A$  be an  $m \times (n+1)$   $(0, 1)$ -matrix not containing an element of  $\mathcal{N}$ . The matrix obtained by removing the  $(n+1)$ -st column from  $A$  is  $m \times n$  matrix. Hence, by the induction assumption, the matrix  $A$  can be transformed as

$$A \sim \left( \begin{array}{ccc|c} F_1 & & O & \mathbf{b}_1 \\ & \ddots & & \vdots \\ O & & F_k & \mathbf{b}_k \\ & & & \mathbf{c} \end{array} \right),$$

where  $F_i$  ( $1 \leq i \leq k$ ) are Ferrers matrices with no zero rows or columns, and  $\mathbf{b}_i$  ( $1 \leq i \leq k$ ) and  $\mathbf{c}$  are  $(0, 1)$ -vectors. If there exist two non-zero

vectors  $\mathbf{b}_i$  and  $\mathbf{b}_j$  ( $i \neq j$ ), then the submatrix  $\begin{pmatrix} F_i & O & \mathbf{b}_i \\ O & F_j & \mathbf{b}_j \end{pmatrix}$  contains a matrix  $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ , and this contradicts the assumption that  $A$  does not contain any element of  $\mathcal{N}$ . Therefore, there is at most one non-zero vector in  $\mathbf{b}_i$  ( $1 \leq i \leq k$ ), and we can set  $\mathbf{b}_1 = \cdots = \mathbf{b}_{k-1} = \mathbf{0}$  without loss of generality.

We consider the two cases where (i)  $\mathbf{c}$  has 1's and (ii)  $\mathbf{c}$  has no 1's.

(i). The case that  $\mathbf{c}$  has 1's.

If the vector  $\mathbf{b}_k$  has both 0's and 1's, then  $\begin{pmatrix} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{pmatrix}$  contains a matrix  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  or  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ , and this contradicts the assumption that  $A$  does not contain an element of  $\mathcal{N}$ . Therefore,  $\mathbf{b}_k = \mathbf{1}$  or  $\mathbf{0}$ . If  $\mathbf{b}_k = \mathbf{1}$ , then

$$\left( \begin{array}{c|c} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{array} \right) \sim \left( \begin{array}{cc|c} \mathbf{1} & F_k & \\ \mathbf{c} & & O \end{array} \right). \quad (5)$$

Because the right-hand side of (5) is a lonesum matrix, the statement holds. If  $\mathbf{b}_k = \mathbf{0}$ , then

$$\left( \begin{array}{c|c} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{array} \right) \sim \left( \begin{array}{cc|c} F_k & \mathbf{0} & \\ \mathbf{c} & & O \end{array} \right). \quad (6)$$

The right-hand side of (6) is lonesum decomposable of order 2, and hence the statement again holds.

(ii). The case that  $\mathbf{c}$  has no 1's.

We have

$$\left( \begin{array}{c|c} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{array} \right) \sim \left( \begin{array}{ccc|c} F_k & \mathbf{b}_k & & \\ & \mathbf{0} & & O \end{array} \right). \quad (7)$$

By Proposition 1.1, if the matrix  $(F_k \mathbf{b}_k)$  is not a lonesum matrix then it contains  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  as a submatrix. Because  $F_k$  has no zero columns, the matrix  $(F_k \mathbf{b}_k)$  also contains  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$  as a submatrix, and this contradicts the assumption that  $A$  does not contain an element of  $\mathcal{N}$ . Therefore, the matrix  $(F_k \mathbf{b}_k)$  is a lonesum matrix and the statement also holds in this case.  $\square$

For a  $(0,1)$ -matrix  $A$ , we define  $\overline{A}$  as the matrix in which the 0 and 1 entries of  $A$  are inverted. If  $A$  is a lonesum matrix, then  $\overline{A}$  is also a lonesum matrix. However, lonesum decomposable matrices do not have this

property. For example, the matrix  $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$  is lonesum decomposable, but  $\overline{V} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{N}$  is not lonesum decomposable. The following theorem determines when a matrix  $A$  satisfies that both  $A$  and  $\overline{A}$  are lonesum decomposable.

**Theorem 2.2.** *Let  $A$  be a  $(0, 1)$ -matrix. Then, the following conditions are equivalent.*

- (i) *Both  $A$  and  $\overline{A}$  are lonesum decomposable.*
- (ii)  *$A$  is a lonesum matrix or  $A \sim \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{pmatrix}$ , where  $\mathbf{1}$  (resp.  $\mathbf{O}$ ) is a matrix whose entries are all 1 (resp. 0).*

*Proof.* It is clear that (ii)  $\Rightarrow$  (i), and we only have to prove that (i)  $\Rightarrow$  (ii). Assume that  $A$  and  $\overline{A}$  are both lonesum decomposable, and let  $k$  be the decomposition order of  $A$ . When  $k = 0$  or  $1$ ,  $A$  is a lonesum matrix. When  $k = 2$ , the matrix  $A$  satisfies that

$$A \sim \begin{pmatrix} L_1 & O \\ O & L_2 \end{pmatrix},$$

where  $L_1$  and  $L_2$  are non-zero lonesum matrices. If  $L_1$  or  $L_2$  has 0's, then the matrix  $\overline{A}$  contains an element of  $\mathcal{N}$  as a submatrix, and  $\overline{A}$  is not lonesum decomposable. Therefore,  $L_1 = \mathbf{1}$  and  $L_2 = \mathbf{1}$ . When  $k \geq 3$ , the matrix

$A$  contains a  $3 \times 3$  submatrix  $W$  satisfying  $W \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . This matrix

contains  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ , and this contradicts the condition that  $\overline{A}$  is lonesum decomposable. As a consequence, either  $A$  is a lonesum matrix or  $A \sim \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{pmatrix}$ .  $\square$

### 3 Generating function of lonesum decomposable matrices

For a positive integer  $k$ , let  $D_k(m, n)$  denote the number of  $m \times n$  lonesum decomposable matrices of decomposition order  $k$ . For simplicity, we set  $D_k(m, 0) = D_k(0, m) = 0$  for  $k \geq 1$  and  $m \geq 0$ , and  $D_0(m, n) = 1$

for  $(m, n) \in \mathbb{Z}_{\geq 0}^2$ . Moreover, we define  $D(m, n) := \sum_{k=0}^{\infty} D_k(m, n)$  for  $(m, n) \in \mathbb{Z}_{\geq 0}^2$ . This means that  $D(m, n)$  is the number of all  $m \times n$  lonesum decomposable matrices. We can see that  $D_k(m, n) = 0$  for  $k > \min(m, n)$  and  $L(m, n) = D_0(m, n) + D_1(m, n)$ . We present tables showing  $D_1(m, n)$ ,  $D_2(m, n)$ , and  $D(m, n)$  at the end of this paper.

The generating functions for  $D_k$  and  $D$  are given as follows:

**Theorem 3.1.** *The following equations hold:*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_k(m, n) \frac{x^m y^n}{m! n!} = \frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \quad (k \geq 0). \quad (8)$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D(m, n) \frac{x^m y^n}{m! n!} = \exp \left( \frac{1}{e^x + e^y - e^{x+y}} + x + y - 1 \right). \quad (9)$$

*Proof.* Let  $\tilde{L}(m, n)$  be the number of  $m \times n$  lonesum matrices with no zero rows or columns. Here, we set  $\tilde{L}(0, 0) = 1$  and  $\tilde{L}(m, 0) = \tilde{L}(0, m) = 0$  for  $m > 0$ . Benyi and Hajnal [3, Theorem 3] mentioned that the generating function of  $\tilde{L}(m, n)$  is given by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tilde{L}(m, n) \frac{x^m y^n}{m! n!} = \frac{1}{e^x + e^y - e^{x+y}}. \quad (10)$$

By definition, it holds that

$$L(m, n) = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \tilde{L}(i, j), \quad (11)$$

and we can also obtain the generating function (3) of  $L(m, n)$  from (10). We note that multiplying the generating function (10) by  $e^{x+y}$  means that it allows the lonesum matrices to have zero columns or zero rows.

Let  $\tilde{D}_k(m, n)$  be the number of  $m \times n$  lonesum decomposable matrices of order  $k$  with no zero rows and columns. We set  $\tilde{D}_0(m, n) = 0$  if  $(m, n) \neq (0, 0)$  and  $= 1$  if  $(m, n) = (0, 0)$ . When  $k = 1$ , we have  $\tilde{D}_1(m, n) = \tilde{L}(m, n)$  if  $(m, n) \neq (0, 0)$  and  $= 0$  if  $(m, n) = (0, 0)$ . Therefore, we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tilde{D}_1(m, n) \frac{x^m y^n}{m! n!} = \frac{1}{e^x + e^y - e^{x+y}} - 1.$$

In general, the generating function of  $\tilde{D}_k$  can be given by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tilde{D}_k(m, n) \frac{x^m y^n}{m! n!} = \frac{1}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \quad (k \geq 0). \quad (12)$$

The generating function of  $D_k$  can be obtained by multiplying (12) by  $e^{x+y}$ , hence we obtain (8). Equation (9) follows immediately from (8).  $\square$

**Remark 3.2.** *Ju and Seo [8] studied generating functions for the number of matrices not including various  $2 \times 2$  matrices. Theorem 3.1 gives a similar result on matrices that do not include the elements of  $\mathcal{N}$ .*

It is known that the numbers  $L(m, n)$  (or the poly-Bernoulli numbers of negative indices) satisfy a recurrence relation (e.g. [2, Prop. 14.3 and 14.4]). Our numbers  $D_k(m, n)$  also satisfy a recurrence relation.

**Proposition 3.3.** *For  $k \geq 1$  and  $m, n \geq 0$ , we have*

$$\begin{aligned} D_k(m+1, n) \\ = D_k(m, n) + \sum_{l=0}^{n-1} \binom{n}{l} ((k-1)D_k(m, l) + D_{k-1}(m, l) + D_k(m, l+1)). \end{aligned}$$

*Proof.* Let  $G_k(x, y) := \frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k$ . By direct calculations, we can verify that

$$\frac{\partial}{\partial x} G_k = G_k + (e^y - 1) \left( (k-1)G_k + G_{k-1} + \frac{\partial}{\partial y} G_k \right). \quad (13)$$

By comparing the coefficients of both sides of (13), we obtain the proposition.  $\square$

To conclude this section, we give a relation between  $D_k(m, n)$  and the poly-Bernoulli polynomials. For any integers  $k_1, \dots, k_r$ , we define the multi-poly Bernoulli(-star) polynomials  $B_{n, \star}^{(k_1, \dots, k_r)}(x)$  by

$$e^{-xt} \frac{\text{Li}_{k_1, \dots, k_r}^{\star}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_{n, \star}^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!}, \quad (14)$$

where

$$\text{Li}_{k_1, \dots, k_r}^{\star}(z) := \sum_{1 \leq m_1 \leq \dots \leq m_r} \frac{z^{m_r}}{m_1^{k_1} \dots m_r^{k_r}}.$$

These polynomials have been introduced by Imatomi [7, §6], but they were defined there with  $e^{-xt}$  replaced by  $e^{xt}$  in (14). When  $r = 1$ , the polynomial  $B_{n, \star}^{(k)}(x)$  coincides with the  $n$ -th poly-Bernoulli polynomial  $B_n^{(k)}(x)$  defined by

$$e^{-xt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}$$

(see e.g., Coppo-Candelpergher [6]).



**Proposition 3.4.** For integers  $k, m, n \geq 0$ , we have

$$D_k(m, n) = \frac{(-1)^k}{k!} \left( 1 + \sum_{i=1}^k \binom{k}{i} (-1)^i B_{n, \star}^{\overbrace{(0, \dots, 0)}^{i-1}, -m} (i-1) \right).$$

*Proof.* For an integer  $i \geq 1$ , we have

$$\begin{aligned} \frac{e^{x+y}}{(e^x + e^y - e^{x+y})^i} &= e^{x+y} \left( \frac{1}{e^y(1 - e^x(1 - e^{-y}))} \right)^i \\ &= e^{x+y} e^{-iy} \sum_{l_1, \dots, l_i \geq 0} e^{(l_1 + \dots + l_i)x} (1 - e^{-y})^{l_1 + \dots + l_i} \\ &= e^{-(i-1)y} \sum_{l_1, \dots, l_i \geq 0} e^{(l_1 + \dots + l_i + 1)x} (1 - e^{-y})^{l_1 + \dots + l_i + 1} \frac{1}{1 - e^{-y}} \\ &= e^{-(i-1)y} \sum_{m=0}^{\infty} \sum_{l_1, \dots, l_i \geq 0} \frac{(1 - e^{-y})^{l_1 + \dots + l_i + 1}}{(l_1 + \dots + l_i + 1)^{-m}} \frac{1}{1 - e^{-y}} \frac{x^m}{m!} \\ &= e^{-(i-1)y} \sum_{m=0}^{\infty} \frac{\text{Li}_{0, \dots, 0, -m}^{\star} (1 - e^{-y})}{1 - e^{-y}} \frac{x^m}{m!} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{n, \star}^{\overbrace{(0, \dots, 0)}^{i-1}, -m} (i-1) \frac{x^m}{m!} \frac{x^n}{n!}. \end{aligned}$$

From this formula and the binomial expansion, we obtain that

$$\begin{aligned} &\frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \\ &= \frac{(-1)^k}{k!} \left( e^{x+y} + \sum_{i=1}^k \binom{k}{i} (-1)^i \frac{e^{x+y}}{(e^x + e^y - e^{x+y})^i} \right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^k}{k!} \left( 1 + \sum_{i=1}^k \binom{k}{i} (-1)^i B_{n, \star}^{\overbrace{(0, \dots, 0)}^{i-1}, -m} (i-1) \right) \frac{x^m}{m!} \frac{x^n}{n!}, \end{aligned}$$

and this proves the proposition.  $\square$

**Remark 3.5.** Kaneko, Sakurai, and Tsumura [11] introduced a sequence  $\mathcal{B}_m^{(-l)}(n)$  as

$$\mathcal{B}_m^{(-l)}(n) := \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} B_m^{(-l-j)}(n) \quad (l, m, n \in \mathbb{Z}_{\geq 0}),$$

where  $\begin{bmatrix} n \\ j \end{bmatrix}$  are the Stirling numbers of the first kind. They proved that this sequence has the following simple generating function:

$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{B}_m^{(-l)}(n) \frac{x^l y^m}{l! m!} = \frac{n! e^{x+y}}{(e^x + e^y - e^{x+y})^{n+1}}. \quad (15)$$

By using this formula, we can also give an expression for  $D_k(m, n)$  in terms of poly-Bernoulli polynomials:

$$D_k(m, n) = \frac{(-1)^k}{k!} \left( 1 + \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{i!} \binom{k}{i+1} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix} B_n^{(-m-j)}(i) \right). \quad (16)$$

## 4 Congruences for $D_k(m, n)$

It is known that the numbers of  $m \times n$  lonesum matrices (or poly-Bernoulli numbers of negative indices) have the following expression:

$$L(m, n) = \sum_{j=0}^{\min(m, n)} (j!)^2 \left\{ \begin{matrix} m+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\},$$

where  $\left\{ \begin{matrix} m \\ j \end{matrix} \right\}$  are the Stirling numbers of the second kind (see e.g., [1] [4]). We note that  $\left\{ \begin{matrix} m \\ j \end{matrix} \right\} = 0$  for  $j > m \geq 1$ . The following proposition says that the numbers  $D_k(m, n)$  also have a similar expression.

**Proposition 4.1.** *For integers  $k \geq 1$  and  $m, n \geq 0$  we have*

$$D_k(m, n) = \frac{1}{k!} \sum_{j=k}^{\min(m, n)} \binom{j-1}{k-1} (j!)^2 \left\{ \begin{matrix} m+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\}. \quad (17)$$

*Proof.* The generating function for  $D_k(m, n)$  can be transformed as

$$\begin{aligned} & \frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \\ &= \frac{e^{x+y}}{k!} \left( \frac{(e^x - 1)(e^y - 1)}{1 - (e^x - 1)(e^y - 1)} \right)^k \\ &= \frac{e^{x+y}}{k!} \sum_{j=k}^{\infty} \binom{j-1}{k-1} (e^x - 1)^j (e^y - 1)^j \\ &= \frac{1}{k!} \sum_{j=k}^{\infty} \binom{j-1}{k-1} \frac{1}{(j+1)^2} \frac{d}{dx} (e^x - 1)^{j+1} \frac{d}{dy} (e^y - 1)^{j+1}. \end{aligned}$$

Because

$$(e^z - 1)^m = m! \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{z^n}{n!},$$

we have

$$\begin{aligned} & \frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \\ &= \frac{1}{k!} \sum_{j=k}^{\infty} \binom{j-1}{k-1} (j!)^2 \sum_{l=j}^{\infty} \sum_{m=j}^{\infty} \left\{ \begin{matrix} l+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} m+1 \\ j+1 \end{matrix} \right\} \frac{x^l}{l!} \frac{y^m}{m!}. \end{aligned}$$

Therefore, we obtain (17).  $\square$

By using this expression, we give some congruences for  $D_k(m, n)$  modulo a prime. We first recall the following lemma in order to prove them. All of the formulas are deduced from the well-known identities

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = \left\{ \begin{matrix} m-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} m-1 \\ k \end{matrix} \right\}, \quad \left\{ \begin{matrix} m \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{n=1}^k (-1)^{k-n} \binom{k}{n} n^m,$$

and we omit their proofs.

**Lemma 4.2.** *Let  $p$  be a prime.*

- (i) *For positive integers  $m$  and  $m'$  with  $m \equiv m' \pmod{p-1}$  and  $0 \leq i \leq p$ , we have  $\left\{ \begin{matrix} m \\ i \end{matrix} \right\} \equiv \left\{ \begin{matrix} m' \\ i \end{matrix} \right\} \pmod{p}$ .*
- (ii)  $\left\{ \begin{matrix} p \\ i \end{matrix} \right\} \equiv 0 \pmod{p}$  for  $2 \leq i \leq p-1$ .
- (iii)  $\left\{ \begin{matrix} m \\ 2 \end{matrix} \right\} = 2^{m-1} - 1$  for  $m \geq 1$ .

**Theorem 4.3.** *Let  $k, m, m', n$ , and  $n'$  be positive integers. For any prime  $p$ , the following congruences hold:*

$$(i) \quad \text{If } k \geq p, \text{ then} \quad D_k(m, n) \equiv 0 \pmod{p}. \quad (18)$$

$$(ii) \quad \text{If } m \equiv m' \text{ and } n \equiv n' \pmod{p-1}, \text{ then}$$

$$D_k(m, n) \equiv D_k(m', n') \pmod{p}. \quad (19)$$

(iii) If  $p > k$ , then

$$D_k(p-1, n) \equiv \begin{cases} 0 & (n \not\equiv 0 \pmod{p-1}) \\ \frac{(-1)^{k-1}}{(k-1)!} & (n \equiv 0 \pmod{p-1}) \end{cases} \pmod{p}. \quad (20)$$

(iv)

$$D_k(p, n) \equiv \begin{cases} 2^n - 1 & (k = 1) \\ 0 & (k \geq 2) \end{cases} \pmod{p}. \quad (21)$$

*Proof.* (i) If  $k \geq p$ , then  $(j!)^2/k! \equiv 0 \pmod{p}$  in (17), and this proves that  $D_k(m, n) \equiv 0 \pmod{p}$ .

(ii) By (i), when  $k \geq p$  both sides of (19) vanish modulo  $p$  and the congruence holds. We may assume that  $p > k$ . By the symmetric property  $D_k(m, n) = D_k(n, m)$ , we only have to show that  $D_k(m+p-1, n) \equiv D_k(m, n) \pmod{p}$ . By Proposition 4.1, we have

$$D_k(m+p-1, n) = \sum_{j=k}^{\min(m+p-1, n)} \binom{j-1}{k-1} \frac{(j!)^2}{k!} \begin{Bmatrix} m+p \\ j+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}. \quad (22)$$

The terms for  $j \geq m+1$  in (22) vanish modulo  $p$ . In fact, if  $m+1 \leq j \leq p-1$  then  $\begin{Bmatrix} m+p \\ j+1 \end{Bmatrix} \equiv \begin{Bmatrix} m+1 \\ j+1 \end{Bmatrix} \equiv 0 \pmod{p}$  by Lemma 4.2 (i), and if  $j \geq p$  then  $j! \equiv 0 \pmod{p}$ . Consequently, we have

$$D_k(m+p-1, n) \equiv \sum_{j=k}^{\min(m, n)} \binom{j-1}{k-1} \frac{(j!)^2}{k!} \begin{Bmatrix} m+1 \\ j+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix} \pmod{p},$$

and this is equal to  $D_k(m, n)$ .

(iii) By (ii), we only have to consider the cases with  $1 \leq n \leq p-1$ . By Proposition 4.1, we have

$$D_k(p-1, n) = \sum_{j=k}^{\min(p-1, n)} \binom{j-1}{k-1} \frac{(j!)^2}{k!} \begin{Bmatrix} p \\ j+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}. \quad (23)$$

If  $n \leq p-2$ , then  $\begin{Bmatrix} p \\ j+1 \end{Bmatrix} \equiv 0 \pmod{p}$  by Lemma 4.2 (ii), and  $D_k(p-1, n) \equiv 0 \pmod{p}$ . If  $n = p-1$ , then only the term for  $j = p-1$  in (23) remains, and

$$D_k(p-1, n) \equiv \binom{p-2}{k-1} \frac{((p-1)!)^2}{k!} \equiv \frac{(-1)^{k-1}}{(k-1)!} \pmod{p}.$$

The final equivalence is derived from the congruence  $\binom{p-2}{k-1} \equiv (-1)^{k-1}k$  and Wilson's theorem, which states that  $(p-1)! \equiv -1 \pmod{p}$ .

(iv) By Proposition 4.1, we have

$$D_k(p, n) = \sum_{j=k}^{\min(p, n)} \binom{j-1}{k-1} \frac{(j!)^2}{k!} \left\{ \begin{matrix} p+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\}. \quad (24)$$

When  $j = p$ , we have  $(j!)^2/k! \equiv 0 \pmod{p}$ . When  $2 \leq j \leq p-1$ , we have  $\left\{ \begin{matrix} p+1 \\ j+1 \end{matrix} \right\} = \left\{ \begin{matrix} p \\ j \end{matrix} \right\} + (j+1) \left\{ \begin{matrix} p \\ j+1 \end{matrix} \right\} \equiv 0 \pmod{p}$  because of Lemma 4.2 (ii). Therefore, the congruence  $D_k(p, n) \equiv 0 \pmod{p}$  holds for  $k \geq 2$ .

If  $k = 1$ , then the term for  $j = 1$  in (24) remains, and  $D_k(p, n) \equiv \left\{ \begin{matrix} p+1 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} n+1 \\ 2 \end{matrix} \right\} = (2^p - 1)(2^n - 1) \equiv 2^n - 1 \pmod{p}$  by Lemma 4.2 (iii) and Fermat's little theorem.

□

Table 1:  $D_1(m, n)$

$m \setminus n$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	3	7	15	31
2	0	3	13	45	145	453
3	0	7	45	229	1065	4717
4	0	15	145	1065	6901	41505
5	0	31	453	4717	41505	329461

Table 2:  $D_2(m, n)$

$m \setminus n$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	2	12	50	180
3	0	0	12	108	660	3420
4	0	0	50	660	5714	40860
5	0	0	180	3420	40860	391500

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Table 3:  $D(m, n)$

$m \backslash n$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	16	58	196	634
3	1	8	58	344	1786	8528
4	1	16	196	1786	13528	90946
5	1	32	634	8528	90446	833432

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